

# NEURAL NETWORKS WITH FINITE-WIDTH ACTION POTENTIALS

Fariel Shafee  
*Department of Physics*  
*Princeton University*  
*Princeton, NJ 08540*  
*USA.\**

The integrate-and-fire type of dynamic neural networks are studied in models with hitherto neglected finite widths to make them more realistic. Pulse shapes of arbitrary widths and shapes are considered and differences in the evolving firing patterns are noted. The convergence to limit cycle is examined both analytically and in simulation experiments. We present a proof of a theorem regarding the timer period of oscillation analogous to that of Hopfield and Herz for nonleaking models with finite width action potentials. It is pointed out that the concept of simultaneity in such networks may be subject to bin widths, i.e. window apertures of cumulative counts, in simulation experiments. The relation to Kolmogorov complexity is briefly discussed. The effects of varying the shape of the action potential and its width are examined in some detail. The roles of the magnitude of a leaking resistance connected to the neurons, a common external current and synaptic currents between neighbors are also examined and found to bear important consequences in the dynamical development of the networks.

PACS numbers: PACS numbers: 07.05.Mh, 84.35.+i, 87.18.Sn

Keywords: neural network; action potential; dynamic memory

## I. INTRODUCTION

Models of neural networks are important to understand biological information processing and storage systems, and also because their relation to many physical systems shed new light on those nonbiological complexes. It is well-known that Hopfield's associative memory networks [1, 2] share many common features with spin glasses [3, 4]. Artificial neural networks may one day pave the way for creating computers which are more akin to organic pattern recognition systems, and hence more efficient in tasks at present not addressable by mechanistic serial processing.

Associative memory networks [5, 6, 7, 8, 9, 10, 11] are usually conceived as static patterns holding memories of inputs, but in recent years there is growing interest in systems which too hold memories of inputs in a dynamic fashion, in the form of phase-locked oscillations. Several different type of models have been presented [14, 15] with different assumptions about their *modus operandi*. The differences may be in the ways the neurons are triggered, in the ways they convey their information to other neurons, in the fashion they are interconnected, and other conjectures about the role of the environment.

Among these models are included a set of relatively simple networks of integrate-and-fire neurons conceived by Hopfield and Herz [2], whose work shows that many

interesting nontrivial predictions regarding the dynamic behavior of such networks could be made from their simple properties. However, in their work, to maintain simplicity, they assumed that the action potential is in the form of a delta function in time, i.e. it is instantaneous. But in some of the models they present, when a neuron fires on attaining threshold from part of the action potential (AP) from a neighbor, the remaining excess charge from the neighbor may remain stored and contributes to the next firing of the receiver, which in a sense mimics the effects of a finite width, without involving quantitative complications that an explicit finite-width model would entail.

A more realistic approach, however, cannot avoid dealing with all the consequences of the finite width of the AP, which is a biological fact and must also be an inevitable feature of most physical realizations. *A priori*, these may be in the form of changed periods, different convergence behavior to limit cycles or modified rates of convergence, different paths to globalization, or its absence altogether etc.

The width introduces its own time scale that can be expected to interact with the time scale of the zero-width period obtained in the H&H models, which in turn depends on the strength of coupling among the neurons and the environmental current. It also introduces the possibility of a more complicated distribution of the action potential current to the receiving neuron, affecting its attainment of threshold, i.e. its period and properties related to convergence to limit cycles.

We have, therefore, investigated the consequences of introducing finite widths into the integrate-and-fire mod-

---

\*Electronic address: fshafee@princeton.edu

els. Where possible we have tried to use analytical methods. Where that turns out to be impossible, we have used simulation. In the next section we first describe different types of integrate-and-fire models to establish the concepts and notation used in this work. In section 3 we show that the convergence to periodic behavior for Class C models can be analytically proved even in the finite width case. In section 4 we present our simulations with class C models using APs of various finite widths and different shapes. In section 5 we do the same with the leaking model A. Finally in section 6 we present our conclusions.

## II. INTEGRATE-AND-FIRE MODELS

Integrate-and-fire neurons are usually considered in four different scenarios [2]. In each case every neuron in the network receives some charge from other neurons in the neighborhood, and also from an environmental current source. This raises the electrostatic potential of the neuron. When the neuron has accumulated sufficient charge and its potential exceeds a threshold, it fires, i.e. it gives forth an AP to the other neurons, which too may be triggered in due course. If all neurons attain the same periodicity, the phase difference among them must become constant, i.e. the system becomes phase-locked. It is also possible for some of the neurons to fire simultaneously (synchronicity) if they reach the limit cycle together.

Let  $u_i(t)$  be the potential of the  $i$ -th neuron at time  $t$ ,  $I_i(t)$  the external current input into this neuron,  $R$  its leaking resistance and  $T_{ij}f_j(t)$  the action current getting into neuron  $i$  from neuron  $j$ . We can then write the current conservation equation as:

$$Cdu_i/dt = -u_i(t)/R + \sum_j T_{ij}f_j(t) + I_i(t) \quad (1)$$

For simplicity with no loss of generality we take the threshold for firing to be  $u = 1$ , by renormalizing  $u$  to the dimensionless quantity  $u/u_{threshold}$ . We can also divide the equation by  $C$  and absorb it in  $R$  and  $I$ , in which case  $R$  actually represents  $CR$ , the decay time constant, and  $I$  represents  $I/(Cu_{threshold})$  which has the dimension of inverse time. In other words, in our notation  $R$  and  $1/I$  will set two different time scales. In most of our work we shall assign the value 1 to  $I$  to represent the scale of an external clock against which all internal dynamics is timed.

In the H&H work the action pulse shape  $f(t)$  is a Dirac delta function, i.e. it is a zero-width normalized pulse:

$$f_j(t) = \delta(t - t_j) \quad (2)$$

where  $t_j$  is the time of the last firing of neuron  $j$ .

The four models differ in the treatment of the leaking as represented by  $R$  and in the way they handle the after-fire potential  $u_i$ .

In Model A,  $R$  is finite and may be taken to be equal to 1. This establishes the time scale  $CR$ . In this model if neuron  $i$  receives a synaptic charge in excess of that required to take the potential to the threshold, then, after firing,  $u_i(t)$  resets to a value corresponding to the excess charge (we omit the implied symbol for summation over neighbors  $j$  for clarity):

$$u_i(t^+) = [u_i(t^-) + \Delta T_{ij}1] + (1 - \Delta)T_{ij} = (1 - \Delta)T_{ij} \quad (3)$$

where

$$\Delta T_{ij} = 1 - u_i(t^-) \quad (4)$$

In Model B, the potential resets to zero after firing, irrespective of the oncoming synaptic current and the previous state of the neuron. It too has a leaking  $R = 1$ . Model C is the version of A that does not leak, i.e.  $R = \infty$ , and Model D is the non-leaking version of B.

One can integrate the model C potential since the last firing at  $t_0$  gives, assuming for simplicity constant external current  $I$ :

$$u_i(t) = u_i(t^+) + \sum_j' T_{ij} + I(t - t_0) \quad (5)$$

where the summation is over only those other neurons  $j$  which have fired since  $t_0$ .

It is convenient to use a simple topology of the network by assuming only nearest neighbor interactions. Then, after phase-locking has been established with period  $P$ , we must have:

$$u(t_0 + P) = 1 = \sum_j T_{ij} + IP \quad (6)$$

So

$$P = (1 - A)/I \quad (7)$$

where for notational convenience, assuming constant  $T_{ij}$ , we get:

$$A = \sum_j T_{ij} = Z\alpha \quad (8)$$

$Z$  being the co-ordination number of the neural lattice, i.e. the number of nearest neighbors. In Model A it is not possible to integrate Eq.1, because the integral depends on the specific times the contributions from the different neighbors are received. This uncertainty results from the leakage, which is proportional to the instantaneous potential.

## III. CONVERGENCE TO PERIODICITY IN THE FINITE WIDTH CASE

When the action potential occupies a finite width we have to use the pulse form factor  $f$  with

$$\int_{t_0}^{t_0 + \Delta t} dt f(t) = 1 \quad (9)$$

Here  $t_o$  is the time of the beginning of a firing and  $\Delta t$  is the width of the action potential and the consequent synaptic current.

As in the case of zero width action potentials, we can show that in the finite width case too the system moves to a limit cycle with phase locking among the neurons. We shall do it for the nonleaking model C. We present the proof in three parts.

I: The time between the beginning of two successive firings of the same neuron cannot be less than  $P = (1 - A)/I$ , where the variables are as defined in Eq. 7 and Eq. 8.

Proof: Let  $i$  be the neuron which has the smallest interval between two firings (there may be other neurons with the same least interval). Let the two firings begin at times  $t_1$  and  $t_2$ . Let us consider any neighbor  $j$  of  $i$ . Since,  $i$  has the least interval between successive firings,  $j$  cannot begin to fire twice within this interval. Hence,  $i$  can receive at most fractional charges from two firings of  $j$  within this interval, of which the beginning of the second firing  $t'_2$  can be within the interval, but not the earlier one  $t'_1$ . So

$$t_2 - t_1 \leq t'_2 - t'_1 \quad (10)$$

i.e.

$$t_2 - t'_2 \leq t_1 - t'_1 \quad (11)$$

The total contribution received from  $j$  by  $i$  within this interval is:

$$\alpha_j = \int_0^{s_2 \Delta t} f(t) dt + \int_{s_1 \Delta t}^{\Delta t} f(t) dt \quad (12)$$

where

$$s_{1,2} = (t_{1,2} - t'_{1,2})/\Delta t. \quad (13)$$

As,  $s_2 \leq s_1$ , we see that

$$\alpha_j \leq \alpha \quad (14)$$

ensuring that the period for  $i$ ,  $P_i \geq P$ .

II: The system converges to a limit cycle for the finite width case:

Proof: The Lyapunov function  $E$  is defined by [2]  $E = -\sum u_i$ . Then its change after an interval  $P$  given by Eq. 7 is

$$\begin{aligned} E(t+P) - E(t) &= \sum_i [-u_i(t+P)] - \sum_i [-u_i(t)] \\ &= \sum_i (-IP - \sum_j T_{ij} \int_t^{t+P} f_j(t) dt \\ &\quad + \sum_i \int_t^{t+P} f_i(t) dt) \\ &= -(1-A)N + (1-A) \sum_i \int_t^{t+P} f_i(t) dt \\ &= -(1-A)[N - \sum_i \int_t^{t+P} f_i(t) dt] \quad (15) \end{aligned}$$

But no neuron can begin to fire twice in an interval of less than  $P$ ; so  $E(t+P) - E(t)$  is nonpositive as in the case of zero width AP. Hence, like the zero width case, we have a convergence to a limit cycle.

III: If there is a periodicity, the period must be  $P = (1 - A)/I$ . Proof: Firstly, by part 1, it cannot be less than  $P$ , because no neuron can ever begin to fire again within an interval less than  $P$ . Indeed if we now assume that a periodicity  $T$  has been established, i.e. all neurons have become phase locked with equal phase difference in all periods, then, if  $t_i$  is the last time neuron  $i$  began to fire:  $u_i(t_i + T) = 1 = \sum_j T_{ij} \int_{t_i}^{t_i+T} f_j(t) dt + IT = A + [(1 - A)/P]T$ .

The last equality follows from the fact that if  $i$  receives a fraction of a pulse from a neighbor  $j$  at the beginning of the cycle, it will receive the rest at the last part of its cycle because of phase lock. i.e.  $\int_t^{t+P} f_j(t) dt = 1$  for all  $j$ . So  $T = P$ .

#### IV. RATE OF CONVERGENCE TO LIMIT CYCLE

An interesting observation can be made for the speed of convergence to the limit cycle with period  $P$ . Let us consider, for simplicity, finite pulses of square shape. Then the contribution from a neighbor  $j$  to a neuron  $i$  will be proportional to the fractional duration of the pulse received by  $i$  within its (variable period) cycle. Again, for simplicity let us consider only one neighbor. Then, if  $P_n$  is the period of the  $n$ -th cycle, and  $f_n$  is the duration of the pulse received by  $i$ , and  $w$  is the width of the pulse, then

$$\begin{aligned} P_{n+1} &= [1(A/w)(w - f_n) - Af_{n+1}/w]/I \\ &= P_0 + A/(wI)[f_n - f_{n+1}] \quad (16) \end{aligned}$$

Now, writing

$$\Delta P_n = P_n - P \quad (17)$$

we get

$$\Delta P_{n+1}/\Delta P_n = A/[Iw + A] \quad (18)$$

We can see that convergence to the limit cycle is a geometric sequence and the rate of convergence depends on the  $A/(Iw + A)$  ratio, which of course is dimensionless in our choice of notation. For fixed  $w$  a higher current would bring in faster convergence. It may seem that for the zero width case we have no convergence at all. But actually in this singular case we cannot split the pulse into two parts as in Eq. 16, so that we must have  $f_n = 1$ , and  $\Delta P_n = 0$  for all  $n$ , i.e. the system goes into phase lock after the completion of the first set of firings. As we pointed out in the beginning, this relation is derived assuming a single neighbor. If  $Z > 1$ , then in general the different neighbors can contribute with different phase differences, which may keep changing until phase lock. Eq. 17 will then need to be modified to account for these differences. If  $r$  of the  $Z$  neighbors contribute fully within a single period, then we have:

$$\Delta P_{n+1}/\Delta P_n = A'/(Iw + A') \quad (19)$$

where we have used the symbol

$$A' = (1 - r/Z)A \quad (20)$$

Therefore, when all the neighbors contribute their full action potential within a single cycle, the period becomes  $P$ . However, the inverse is not true, i.e. it is possible to distribute the charge from a single AP to succeeding cycles of the neighbor and the whole system can still be in total phase-lock with period  $P$ , because the R.H.S. of Eq. 19 becomes indeterminate. In the case of pulses of arbitrary shapes it is easy to see that we get:

$$\Delta P_{n+1} = \sum_j (\alpha_j/I) + \int_{P_{n+1}}^{P_n} f(t)dt \quad (21)$$

where the sum runs over only those neighbors which do not yet deliver their full synaptic current in a single cycle. This is an integral equation depending on the shape function  $f(t)$ . Despite the arbitrariness of  $f(t)$  we can see that if  $\Delta P_{n+1} \rightarrow \Delta P_n$ , then both must approach zero, i.e. the sequence must be convergent to the limit cycle. If  $\Delta P_{n+1}/\Delta P_n \rightarrow r < 1$ , then too the convergence is obvious. If as a first approximation we replace the arbitrary pulse by a square one of same height but a width giving unit area, then we would expect a geometric convergence similar to Eq.18- Eq. 20. The details of the pulse shape may produce smaller perturbations on the rate of convergence without affecting the general pattern.

## V. NONLEAKING NETWORKS WITH FINITE WIDTH A.P.

We simulate a square lattice with periodic boundary conditions to reduce finite size effects. It has been noted that a 40 X 40 lattice is sufficient to demonstrate all the

important characteristics. At each step the neurons are updated according to the charge they receive from their neighbors and the external current during the time loop and then, if the potential reaches threshold, it fires. The effect of the finite width of the action potential can be two-fold: (1) reduction of  $u$  over a number of time loops corresponding to the width of the AP after a neuron begins to fire, and (2) the arrival of the synaptic current  $T_{ij}f(t)$  over a number of time loops according to the shape of the packet  $f(t)$ . We do not expect any role for ectopic pulses resulting from recharging to threshold before the previous firing is completed, though the synaptic current arriving at a neuron while it discharges will be retained for the next firing. Hence, in the nonleaking models it is unnecessary to model the actual shape of the fall of  $u$ . The results will be indistinguishable from those of a zero-width model in such models, though the situation in leaking models will be quite different because the leakage will depend on the falling potential. The distribution of the synaptic current over multiple time loops, however, will carry the nontrivial differential characteristics of finite width action potentials in all models. In this section we consider only a nonleaking model. We have in our algorithm provision for generating pulses of various shapes in the form of discretized  $f(t_i)$  over the time loops during which the action pulse works. Arbitrary shapes can also be given as input. However, we have only done the simulations with square and isosceles triangular shapes for simplicity.

### A. Effect of Pulse Shape:

In Fig.1 and Fig. 2 we present our simulations with  $A = 0.96$  (i.e.  $T_{ij} = 0.24$ ),  $I = 1$  and  $width = 0.01$  (which is  $1/4$  of  $P$ ) using first a square pulse (Fig.1), and then a triangular one (Fig.2). We note that indeed in both cases the period 0.04 is equal to  $P[(1 - A)/I]$ , as proved earlier. However, there is a slight difference between them in their rate of convergence to the limit cycle; the square pulses almost converge after  $t = 4$ , whereas for the triangular ones a similar level of convergence happens after  $t = 6$ . This can be expected on account of the simplicity of the square pulse allowing it to adapt itself more easily for phase-locking.

### B. Convergence Rate and the Value of A:

In Fig.3 and Fig.4 we show the effect of changing the value  $A$ , keeping the same ratio of the width with the corresponding period  $P$ . Fig.3 is for a square pulse and Fig.4 is for a triangular one. Of course varying  $A$  changes the period. But we have also seen that the rate of convergence depends on the relative magnitudes of  $A$ ,  $I$  and  $w$  according to Eq.18- Eq. 21. A higher contribution from the neighboring neurons than the constant external driver seems to delay convergence, which may be

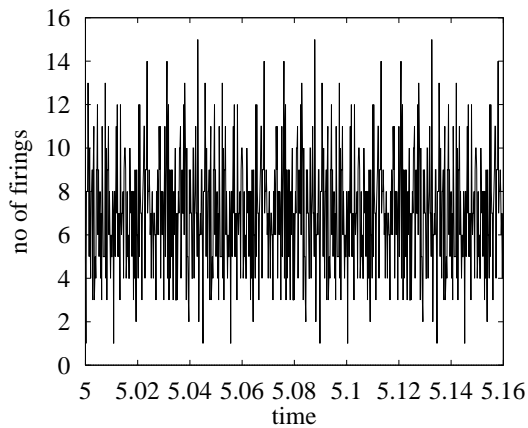


FIG. 1: Square pulse with width  $w = 0.01$ ,  $A = 0.96$ ,  $I = 1$ .

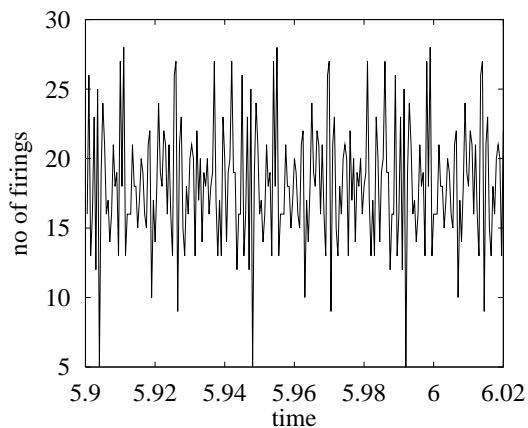


FIG. 2: same as Fig. 1, but with a triangular pulse.

expected on account of the more intricate interactions in phase adaptation of the neighboring neurons than in their susceptibility for acting cohesively with a common external driver. For  $A = 0.24$ , i.e.  $\alpha = 0.06$ , we get practically immediate convergence after all the neurons have fired for the first time. This feature is common for both shapes of pulses.

### C. Effect of Region of Initial Excitation:

The network can be made to receive an initial excitation at the periphery, or throughout the network. In Fig.5 we repeat the experiment of Fig.2, with the same parameters, but with only the peripheral neurons initially excited. We see that the general pattern remains the same, and cannot be distinguished from the whole net excitation of Fig. 2. However, in Fig. 6 we have done the same with the smaller  $A(= 0.24)$  case, first studied with whole net excitation in Fig.4. There is remarkable synchronicity in this case, the great majority of the neurons firing together periodically. We can understand what is happening here if we picture the initial development of

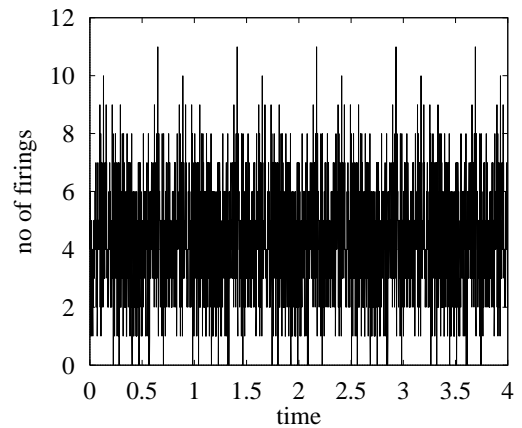


FIG. 3: Square pulse, with  $A = 0.24$ ,  $I = 1$ ,  $w = 0.2$

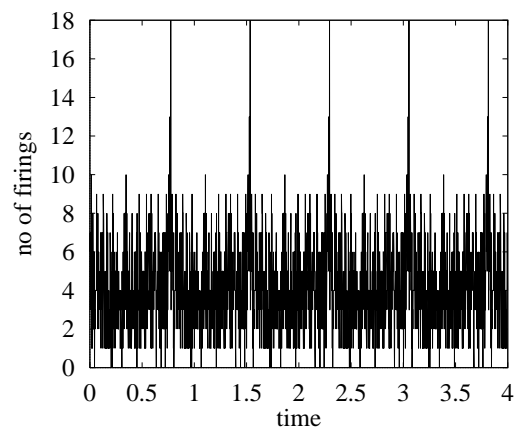


FIG. 4: same as Fig. 3, but with a triangular A.P. pulse.

the system. All the inner neurons are initially uncharged and they do not receive pulses from neighbors until their first firing at  $t = 1$ , which is caused simultaneously by the common driver  $I$ . Henceforth, all these inner neurons remain phase locked. By contrast for  $A = 0.96$ , the external current has little role in charging up the neurons, and hence the neurons get phase-locked only after their mutual interactions, making the situation similar to the whole net excitation.

## VI. SYNCHRONICITY, BIN SIZE AND DYNAMIC ENTROPY:

While the synchronicity just described, induced by the common external agent, may indeed be genuine, the other synchronicities may be more suspect. Since in simulation experiments the time loops must occur after a small but finite duration, the procedure puts many events of different exact times in the same bin. This is revealed if we change the bin size in the experiment. Neurons which appear in the same bin seem to split up when the resolution is increased. In Fig.7 we show the results of

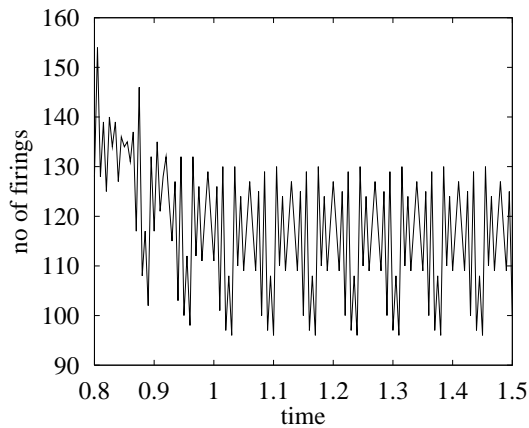


FIG. 5: As Fig. 2, but with peripheral initial excitation.

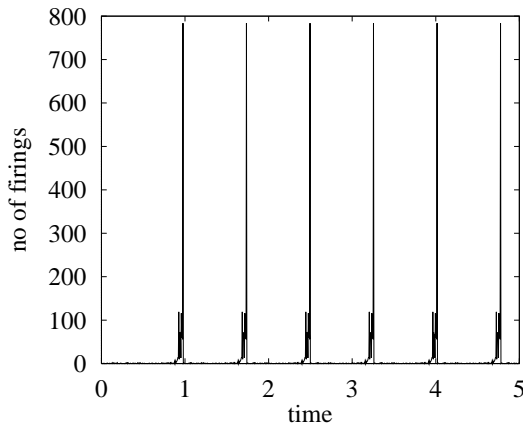


FIG. 6: As Fig. 4, but with peripheral initial excitation.

lumping 10 loops of Fig.2 into one loop. Apart from the proportionately higher bin counts, convergence also appears to arrive faster, which of course is only an artifact.

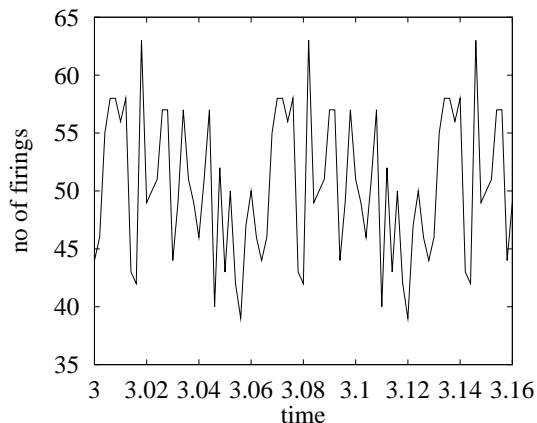


FIG. 7: As Fig. 2, but with a time step 10 times greater.

Because of the built-in time scales related to charging rate, affinities with nearest neighbors, and the width of the action pulses, it is not expected that such neural

networks would show self-similarity, i.e. a fractal structure. It is obvious that if the time bin size is made small enough, the amplitude, i.e. the number of firings per bin, would normally be reduced to a sequence of units and zeros, though, because of the discrete nature of the upgradings in a computer simulation, there may be occasional coincidences. A two-letter alphabet can then define any time sequence of an arbitrary length.

In a time series Kolmogorov-Sinai entropy [12, 13] is occasionally used as a characteristic measure of complexity of the series. It is a measure of the size of the algorithm needed to reproduce the series. A completely random sequence has infinite K-S entropy, a chaotic system has finite entropy, and a completely periodic system has zero entropy. The latter follows from the fact that in amplitude space the trajectory of the system follows the same path repeatedly; for a sequence of  $N$  such steps:

$$K_N = - \sum_{i_0, i_1, \dots, i_N} p_{i_0, i_1, \dots, i_N} \log(p_{i_0, i_1, \dots, i_N}) \quad (22)$$

and hence for a phase-locked periodic system, only a particular sequence of amplitudes  $\{i_0, i_1, \dots, i_N\}$  will have probability 1, and all other sequences will have probability zero, giving zero  $K_N$  and hence zero K-S entropy. The K-S entropy is the difference

$$S_{KS} = \lim_{\epsilon \rightarrow 0, N \rightarrow \infty} (K_{N+1} - K_N) \quad (23)$$

where  $\epsilon$  is the size of size of a unique state in the embedding phase space. In this case since we have a discrete phase space with integral amplitudes, and hence the transition  $\epsilon \rightarrow 0$  is also discrete. A chaotic or random case might produce problems in such a case, but not in a purely deterministic situation as this one.

Entropy would be null irrespective of the complexity of the pattern, and simplifying the pattern by summing bins together in any well-defined nonrandom fashion would not change the entropy from its null value.

For the initial time domain containing the transition to the phase-locking state, since the system is fully deterministic, despite the apparent random pattern of the firings it follows a unique trajectory, given a unique initial state, and hence there is only point in  $N$ -dimensional or in  $N + 1$ -dimensional embedding space with nonzero probability, again giving zero entropy.

## VII. LEAKING NETWORKS WITH FINITE WIDTH A.P.

In Fig.8 we have shown the results of simulating a neural network similar to that of Fig. 2, but with a finite value of the leaking resistance  $R$ . Compared to Fig. 2 we see a higher time period  $T = 0.047$ , though in this case the convergence seems to have taken place much earlier at  $t = 1.7$ . For  $R = 2$ , we get  $T = 0.069$ , and

the limit cycle seems to appear only around  $t = 3.5$ . The smaller resistance of course dampens the charging more and this results in the higher time period. It also effectively reduces the effect of the charging external current  $I$  and, as we have commented earlier, this weakening of a common driver increases the role of the complex phase adjustments of the neighbors, which in turn delays the convergence to the limit cycle. If we integrate Eq. 1 assuming like H & H that all the synaptic current from the neighbors arrive at the beginning of the cycle, and that we have narrow widths as a first approximation, but keeping  $R$  as a free time scale, we get

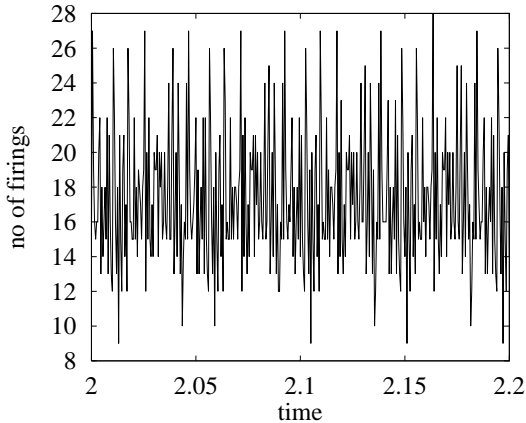


FIG. 8: As Fig. 2, but with leaking neurons with  $R = 10$ .

$$T = R \log[(IR - A)/(IR - 1)] \quad (24)$$

Again, this is derived assuming that each neuron gets the full synaptic current from the neighbors at the very beginning of the cycle and a narrow width as in H&H. We notice that though  $IR$  is a dimensionless quantity here, the period  $T$  is controlled by the scale of  $R$  if  $IR$  is constant, i.e., if we vary both  $I$  and  $R$  keeping  $IR$  constant, then

$$T_1/T_2 = R_1/R_2 \quad (25)$$

So, for  $I = 1$  which we have used throughout as an external time scale reference, making  $R = 1$  should give a singularity. Indeed when we tried to simulate a network with  $I = 1$  and  $R = 1$ , not a single neuron fired. However, if we change  $R$  to a slightly higher value, e.g.  $R = 1.0001$ , other parameters remaining identical, we see a very interesting phenomenon. The neurons now get sufficient accumulated charge from their initial random charges, the external current and the synaptic current from neighbors to defeat the damping of the potential due to leakage and reach firing threshold. However, as the damping is severe near the critical value  $R=1$ , the differentiation of the initial phases becomes relatively unimportant and many neurons move towards synchronicity.

TABLE I: Time period for leaking networks

$R$	1.2	1.5	2.0	5.0	10.0
$T_{formula}$	0.22	0.11	0.078	0.050	0.044
$T_{sim}[w = 0.001]$	0.095	0.078	0.068	0.051	0.048
$T_{sim}[w = 0.002]$	0.21	0.11	0.077	0.049	0.045

In Fig. 9 we see the first bunch of firings, with some dispersion, whereas in Fig.10, which gives a more extended picture, the synchronicity appears to become more established in successive firings. This is in contrast to the case of weaker damping (Fig.8) described above, where periodicity establishes phase-locking, but not synchronicity. Eq. 24 seems to be more accurate for large  $R$  with small  $T$ , than for small  $R$  near criticality (Table 1).

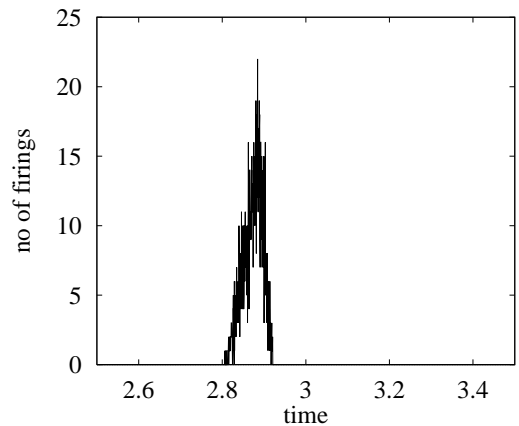


FIG. 9: First firing of a net of leaking neurons, with a resistance just above the critical value of  $R = 1$ .

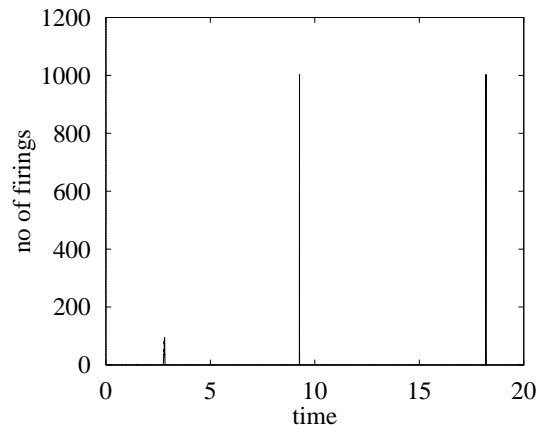


FIG. 10: As above, showing successive peaks with growing synchronicity. The first firings shown in detail in the previous figure are the small clump at around  $t = 2.8$ .

It is understandable that for large periods the discrepancy between Eq. 24 and the results of the simulation increase. The neighboring neurons can then distribute

their synaptic contributions over a larger duration, and any signal arriving late is attenuated by  $R$  less than those arriving at the beginning of the cycle, which we have assumed. So, for large  $T$ , the effective  $A$  in Eq. 24 is higher than the real  $A$ , which decreases the period to some extent. We also note that if the width of the synaptic current pulse is changed there may be a change in the period, because a wider pulse is more liable to get attenuated by the damping  $R$ . The last row of the table presents the results of simulating with  $w = 0.002$ , which gives almost perfect agreement with Eq. 24.

## VIII. CONCLUSIONS

In this work we find that many of the results of an over-simple zero-width integrate-and-fire neural network are valid for more realistic finite-width models. We have proved analytically that even with a finite width, the system converges to a limit cycle with a period identical to the zero-width case. However, the problem of convergence is more complicated for the case of finite width AP spikes, and we have shown that probably only asymptotic convergence ensues as a geometric sequence,

which depends on the parameters of the model, i.e. the width, the external current and the synaptic coupling with the neighbors. The shape of the pulse does not seem to be the determining factor in controlling convergence, though it might have some perturbative modifications. The question of synchronicity seems to be complicated by the inevitable finiteness of the time steps in a simulation and opens up the possibility of investigating any fractal properties associated with such networks. The radical difference between peripheral and whole net excitation also seems interesting and worth further investigation. Stochastic and mean field approximations may give analytic results where exact methods may be limited, as in the case of leaking neurons, though the simple formula Eq. 24 for the period, derived with drastic assumptions give remarkably accurate predictions. The transition to sharp synchronicity in almost critically damped networks may therefore be amenable to more precise analysis.

## Acknowledgments

The author would like to thank Professor J.H. Hopfield for useful discussions.

- 
- [1] J.J. Hopfield, Proc. Natl. Acad. Sci. **79**, 2554 (1982)
  - [2] J.J. Hopfield and A.V.M. Herz, *ibid* **92**, 6655 (1995)
  - [3] P.W. Anderson, Mat. Res. Bull. **5**, 549 (1970)
  - [4] D.L. Stein(ed.), *Spin Glasses and Biology* (World Scientific, 1992)
  - [5] D.J. Amit, H. Gutfreund and H. Sompolinsky, Phys. Rev. Lett. **55**, 1530 (1985)
  - [6] B. Derida, E. Gardner and A. Zippelius, Europhys. Lett. **4**, 167 (1987)
  - [7] H. Gutfreund and M. Mezard, Phys. Rev. Lett. **61**, 235 (1988)
  - [8] H. Sompolinsky and I. Kantner, Phys. Rev. Lett. **57**, 2861 (1986)
  - [9] D. Horn, Physica A **200**, 594 (1993)
  - [10] F. Zertuche, R. Lopes-Pena and H. Waelbroech, J. Phys. A **27**, 5879 (1994)
  - [11] T.L.H. Watkin and D. Sherrington, J. Phys. A **24**, 5427 (1991)
  - [12] A. Kolmogorov, Dokl. Acad. Nauk. SSSR **124**, 754 (1959)
  - [13] Ya. G. Sinai, Dokl. Acad. Nauk. SSSR **124**, 768 (1959)
  - [14] F.C. Hoppensteadt and E.M. Izhikevich, Biological Cybernetics **75**, 117 (1997)
  - [15] T. Aonishi, Osaka University preprint, arXiv:cond-mat/9808121 (1998)